

The Hilbert-Schmidt analyticity associated with infinite-dimensional unitary groups

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Abstract. The article is devoted to the problem of Hilbert-Schmidt type analytic extensions in Hardy spaces over the infinite-dimensional unitary group endowed with an invariant probability measure. Reproducing kernels of Hardy spaces, integral formulas of analytic extensions and their boundary values are considered.

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1. Introduction

The paper deals with the problem of Hilbert-Schmidt type analytic extensions in the Hardy space H_χ^2 of complex functions over the infinite-dimensional group $U(\infty) = \bigcup \{U(m) : m \in \mathbb{N}\}$ endowed with an invariant probability measure χ where $U(m)$ are subgroups of unitary $m \times m$ -matrices. The measure χ is defined as a projective limit $\chi = \varprojlim \chi_m$ of the Haar probability measures χ_m on $U(m)$. Moreover, χ is supported by a projective limit $\mathfrak{U} = \varprojlim U(m)$ and is invariant under the right action of $U^2(\infty) := U(\infty) \times U(\infty)$ on \mathfrak{U} .

A goal of this work is to find integral formulas for Hilbert-Schmidt analytic extensions of functions from H_χ^2 and to describe their radial boundary values on the open unit ball in a Hilbert space \mathbb{E} where $U(\infty)$ acts irreducibly.

The measure χ on \mathfrak{U} was described by G. Olshanski [13], Y. Neretin [12]. The notion \mathfrak{U} is related to D. Pickrell's space of virtual Grassmannian [16]. Hardy spaces in infinite-dimensional settings were discussed in the works of B. Cole, T.W. Gamelin [5], B. Ørsted, K.H. Neeb [14]. Spaces of analytic functions of Hilbert-Schmidt holomorphy types were considered by T.A.W. Dwyer III [6], H. Petersson [15]. More general classes of analytic functions associated

with coherent sequences of polynomial ideals were described by D. Carando, V. Dimant, S. Muro [4]. Integral formulas for analytic functions employing Wiener measures on infinite-dimensional Banach spaces were suggested by D. Pinasco, I. Zalduendo [17].

Note that spaces of integrable functions with respect to invariant measures over infinite-dimensional groups have been widely applied in stochastic processes [2, 3], as well as in other areas.

This paper presents the following results. In Theorem 3.2, we describe an orthogonal basis in the Hardy space H_χ^2 indexed by means of Yang diagrams, consisting of χ -essentially bounded functions. Using this basis, in Theorem 4.2 the reproducing kernel of H_χ^2 is calculated. It also allows us to define an antilinear isometric isomorphism \mathcal{J} between H_χ^2 and the symmetric Fock space Γ generated by E . This isomorphism equips H_χ^2 with a suitable infinite-dimensional analytic structure. By means of \mathcal{J} , we establish in Theorem 6.2 an integral formula for Hilbert-Schmidt analytic extensions of functions from H_χ^2 on the open unit ball $B \subset E$. The radial boundary values of these analytic extensions are described in Theorem 7.1.

2. Background on invariant measure

Let $U(m)$ ($m \in \mathbb{N}$) be the group of unitary $(m \times m)$ -matrices. We endow $U(\infty) = \bigcup U(m)$ with the inductive topology under every continuous inclusion $U(m) \hookrightarrow U(\infty)$ which assigns to any $u_m \in U(m)$ the matrix $\begin{bmatrix} u_m & 0 \\ 0 & \mathbb{1} \end{bmatrix} \in U(\infty)$. The right action over $U(\infty)$ is defined via

$$u.g = w^{-1}uv, \quad u \in U(\infty), \quad g = (v, w) \in U^2(\infty) \quad (2.1)$$

(the right action over $U(m)$ is defined similarly with $u \in U(m)$ and $g = (v, w) \in U^2(m)$ where $U^2(m) := U(m) \times U(m)$).

Following [12, 13], every $u_m \in U(m)$ with $m > 1$ can be written as $u_m = \begin{bmatrix} z_{m-1} & a \\ b & t \end{bmatrix}$ so that z_{m-1} is a $(m-1) \times (m-1)$ -matrix and $t \in \mathbb{C}$. It was proven that the Livšic-type mapping (which is not a group homomorphism)

$$\pi_{m-1}^m: u_m \mapsto u_{m-1} := \begin{cases} z_{m-1} - [a(1+t)^{-1}b] & : t \neq -1 \\ z_{m-1} & : t = -1 \end{cases} \quad (2.2)$$

from $U(m)$ onto $U(m-1)$ is Borel and surjective.

Consider the projective limit $\mathfrak{U} = \varprojlim U(m)$ taken with respect to π_{m-1}^m . The embedding $\rho: U(\infty) \hookrightarrow \mathfrak{U}$ assigns to every $u_m \in U(m)$ the stabilized sequence $u = (u_k)_{k \in \mathbb{N}}$ (see [13, n.4]) so that

$$\rho: U(m) \ni u_m \mapsto (u_k) \in \mathfrak{U}, \quad u_k = \begin{cases} \pi_k^m(u_m) & : k < m, \\ u_m & : k = m, \\ \begin{bmatrix} u_m & 0 \\ 0 & 1 \end{bmatrix} & : k > m \end{cases} \quad (2.3)$$

where the projections $\pi_m: \mathfrak{U} \ni u \longrightarrow u_m \in U(m)$ such that $\pi_{m-1}^m \circ \pi_m = \pi_{m-1}$ are surjective and $\pi_k^m := \pi_k^{k+1} \circ \dots \circ \pi_{m-1}^m$ for $k < m$. Using (2.1), the right action of $U^2(\infty)$ over \mathfrak{U} can be defined as

$$\pi_m(u.g) = w^{-1}\pi_m(u)v, \quad u \in \mathfrak{U} \quad (2.4)$$

where m is so large that $g = (v, w) \in U^2(m)$ (see [13, Def 4.5]).

We endow every group $U(m)$ with the probability Haar measure χ_m . It is known [12, Thm 1.6] that the pushforwards of χ_m to $U(m-1)$ under π_{m-1}^m is the probability Haar measure χ_{m-1} on $U(m)$. Let $U'(m)$ be the subset in $U(m)$ of matrices which do not have $\{-1\}$ as an eigenvalue. Then $U'(m)$ is open in $U(m)$ and $U(m) \setminus U'(m)$ is χ_m -negligible. Moreover, the restriction $\pi_{m-1}^m: U'(m) \longrightarrow U'(m-1)$ is continuous and surjective [13, Lem. 3.11].

Following [13, Lem. 4.8], [12, n.3.1], via of the Kolmogorov consistency theorem we uniquely define on \mathfrak{U} the probability measure χ which is the projective limit under the mapping (2.2), i.e., we put

$$\chi = \varprojlim \chi_m \quad \text{with} \quad \chi_m = \chi \circ \pi_m^{-1} \quad \text{for all} \quad m \in \mathbb{N}. \quad (2.5)$$

If $\mathfrak{U}' = \varprojlim U'(m)$ is the projective limit with respect to $\pi_{m-1}^m|_{U'(m)}$ then $\mathfrak{U} \setminus \mathfrak{U}'$ is χ -negligible, because χ_m is zero on $U(m) \setminus U'(m)$ for any m .

A complex-valued function on \mathfrak{U} is called cylindrical if it has the form $f = f_m \circ \pi_m$ for a certain $m \in \mathbb{N}$ and a complex function f_m on $U(m)$ [13, Def. 4.5]. By L_χ^∞ we denote the closed linear hull of all cylindrical χ -essentially bounded Borel functions endowed with the norm $\|f\|_{L_\chi^\infty} = \text{ess sup}_{u \in \mathfrak{U}} |f(u)|$.

The measure (2.5) is a probability measure and is $U^2(\infty)$ -invariant under the right actions (2.4) over \mathfrak{U} [12, Prop. 3.2]. Moreover, this measure is Radon so that

$$\int_{\mathfrak{U}} f(u.g) d\chi(u) = \int_{\mathfrak{U}} f(u) d\chi(u), \quad g \in U^2(\infty), \quad f \in L_\chi^\infty \quad (2.6)$$

and it satisfies the property: $(\chi \circ \pi_m^{-1})(K) = \chi_m(K)$ for any compact set K in $U(m)$ [11, Lem. 1]. Using the invariance property (2.6) and the Fubini theorem (see [11, Lem. 2]), we obtain

$$\int_{\mathfrak{U}} f d\chi = \int_{\mathfrak{U}} d\chi(u) \int_{U^2(m)} f(u.g) d(\chi_m \otimes \chi_m)(g), \quad (2.7)$$

$$\int_{\mathfrak{U}} f d\chi = \frac{1}{2\pi} \int_{\mathfrak{U}} d\chi(u) \int_{-\pi}^{\pi} f[\exp(i\vartheta)u] d\vartheta \quad (2.8)$$

for all $f \in L_\chi^\infty$. The closed linear hull of cylindrical complex functions endowed with the norm $\|f\|_{L_\chi^2} = (\int_{\mathfrak{U}} |f|^2 d\chi)^{1/2}$ is denoted by L_χ^2 . It is clear that $L_\chi^\infty \hookrightarrow L_\chi^2$ and $\|f\|_{L_\chi^2} \leq \|f\|_{L_\chi^\infty}$ for all $f \in L_\chi^\infty$.

3. Hardy spaces

Throughout the paper \mathbf{E} is a separable complex Hilbert space with an orthonormal basis $\{\mathbf{e}_k : k \in \mathbb{N}\}$, scalar product $\langle \cdot | \cdot \rangle$ and norm $\|\cdot\| = \langle \cdot | \cdot \rangle^{1/2}$.

So, for any element $x \in \mathbf{E}$ the following Fourier decomposition holds,

$$x = \sum \mathbf{e}_k \hat{x}_k, \quad \hat{x}_k = \langle x \mid \mathbf{e}_k \rangle. \quad (3.1)$$

In what follows, let $\mathbf{B} = \{x \in \mathbf{E} : \|x\| < 1\}$ and $\mathbf{S} = \{x \in \mathbf{E} : \|x\| = 1\}$.

Let $\mathbf{E}^{\otimes n}$ be the complete n th tensor power of \mathbf{E} endowed with the scalar product and norm

$$\langle \psi \mid \phi \rangle = \langle x_1 \mid y_1 \rangle \dots \langle x_n \mid y_n \rangle, \quad \|\psi\| = \langle \psi \mid \psi \rangle^{1/2}$$

for all $\psi = x_1 \otimes \dots \otimes x_n, \phi = y_1 \otimes \dots \otimes y_n \in \mathbf{E}^{\otimes n}$ with $x_i, y_i \in \mathbf{E}$ ($i = 1, \dots, n$). As $\sigma : \{1, \dots, n\} \mapsto \{\sigma(1), \dots, \sigma(n)\}$ runs through all n -elements permutations, the symmetric complete n th tensor power $\mathbf{E}^{\odot n}$ is defined to be a codomain of the orthogonal projector

$$\mathbf{E}^{\otimes n} \ni \psi \mapsto x_1 \odot \dots \odot x_n := \frac{1}{n!} \sum_{\sigma} x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(n)} \in \mathbf{E}^{\odot n}.$$

Note that $x^{\otimes n} = x \otimes \dots \otimes x = x \odot \dots \odot x = x^{\odot n}$. Put $\mathbf{E}^{\otimes 0} = \mathbf{E}^{\odot 0} = \mathbb{C}$.

Let $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{N}^m$ be a partition of an integer $n \in \mathbb{N}$ with $m \leq n$ and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m > 0$, i.e., $|\lambda| = n$ where $|\lambda| := \lambda_1 + \dots + \lambda_m$. We identify partitions with Young diagrams. By $\ell(\lambda) = m$ we denote the length of λ defined as the number of rows in λ . Let \mathbb{Y} denote all Young diagrams and $\mathbb{Y}_n := \{\lambda \in \mathbb{Y} : |\lambda| = n\}$. Assume that \mathbb{Y} includes the empty partition $\emptyset = (0, 0, \dots)$.

An orthogonal basis in $\mathbf{E}^{\odot n}$ is formed by the system of symmetric tensor products (see e.g. [1, Sec. 2.2.2])

$$\mathbf{e}^{\odot \mathbb{Y}_n} = \bigcup_{\lambda \in \mathbb{Y}_n} \{ \mathbf{e}_i^{\odot \lambda} := \mathbf{e}_{i_1}^{\otimes \lambda_1} \odot \dots \odot \mathbf{e}_{i_m}^{\otimes \lambda_m} : i \in \mathbb{N}_*^m, m = \ell(\lambda) \}, \quad \mathbf{e}_i^{\odot \emptyset} = 1$$

where $\mathbb{N}_*^m := \{i = (i_1, \dots, i_m) \in \mathbb{N}^m : i_j \neq i_k, \forall j \neq k\}$. As is well known,

$$\|\mathbf{e}_i^{\odot \lambda}\|^2 = \frac{\lambda!}{n!}, \quad \lambda! := \lambda_1! \cdot \dots \cdot \lambda_m!. \quad (3.2)$$

In what follows, we will use the fact that for every $\psi \in \mathbf{E}^{\odot n}$ one can uniquely define the so-called *Hilbert-Schmidt n -homogenous polynomial*

$$\psi^*(x) := \langle x^{\otimes n} \mid \psi \rangle, \quad x \in \mathbf{E}.$$

In fact, the polarization formula for symmetric tensor products (see [8, 1.5])

$$z_1 \odot \dots \odot z_n = \frac{1}{2^n n!} \sum_{\theta_1, \dots, \theta_n = \pm 1} \theta_1 \dots \theta_n x^{\otimes n}, \quad x = \sum_{k=1}^n \theta_k z_k \quad (3.3)$$

($z_1, \dots, z_n \in \mathbf{E}$) implies that the n -homogenous polynomial $\langle x^{\otimes n} \mid \psi \rangle$ is uniquely defined by ψ , because the set $z_1 \odot \dots \odot z_n$ is total in $\mathbf{E}^{\odot n}$.

Using the embedding (2.3), we define the \mathbf{E} -valued mapping

$$\zeta : \mathfrak{U} \ni u \mapsto \rho^{-1}(u) \mathbf{e}_1$$

which do not depend on the choice of \mathbf{e}_1 in

$$\mathbf{S}(\infty) := \{\zeta(u) : u \in \mathfrak{U}\} = \bigcup \{\mathbf{S}(m) : m \in \mathbb{N}\}$$

where $S(m)$ is the m -dimensional unit sphere. In fact, for each stabilized sequence $u = (u_k) \in \mathfrak{U}$ there exists an index m such that $\rho^{-1}(u)\mathbf{e}_1 = u_k\mathbf{e}_1$ belongs to $S(m)$ for all $k \geq m$. On the other hand, for each $\mathbf{e} \in S(k)$ there exists $v \in U(k)$ such that $v\mathbf{e} = \mathbf{e}_1$. Defining $u.g \in \mathfrak{U}$ with $g = (1, v) \in U^2(k)$ by means of (2.3)-(2.4), we have $\rho^{-1}(u.g)\mathbf{e} = \pi_k(u.g)\mathbf{e} = \pi_k(u)\mathbf{e}_1 = \rho^{-1}(u)\mathbf{e}_1$.

Consider the following system of cylindrical Borel functions

$$\varepsilon_k(u) := \langle \zeta(u) \mid \mathbf{e}_k \rangle, \quad k \in \mathbb{N}$$

where $\varepsilon_k := \mathbf{e}_k^* \circ \zeta$. Using ζ , we may define the $E^{\odot n}$ -valued Borel mapping

$$\zeta^{\otimes n}: \mathfrak{U} \ni u \mapsto \underbrace{\zeta(u) \otimes \dots \otimes \zeta(u)}_n, \quad \zeta^{\otimes 0} \equiv 1.$$

The following assertion, which is a consequence of the polarization formula (3.3), is proved in [11, Lem. 3].

Lemma 3.1. *The equality $S(\infty) = \{\zeta(u): u \in \mathfrak{U}'\}$ holds. As a consequence, to every $\psi \in E_i^{\odot n}$ there uniquely corresponds the function in L_X^∞*

$$\psi_\zeta(u) := \langle \zeta^{\otimes n}(u) \mid \psi \rangle, \quad u \in \mathfrak{U}$$

given by continuous restriction to \mathfrak{U}' . In particular, to every $\mathbf{e}_i^{\odot \lambda} \in \mathbf{e}^{\odot \mathbb{Y}_n}$ there corresponds in L_X^∞ the cylindrical function in the variable $u \in \mathfrak{U}$,

$$\varepsilon_i^\lambda(u) := \langle \zeta^{\otimes n}(u) \mid \mathbf{e}_i^{\odot \lambda} \rangle = \prod_{k=1}^{\ell(\lambda)} \langle \zeta(u) \mid \mathbf{e}_{i_k} \rangle^{\lambda_k}. \quad (3.4)$$

Lemma 3.1 straightforwardly implies that the system $\mathbf{e}^{\odot \mathbb{Y}} := \bigcup \mathbf{e}^{\odot \mathbb{Y}_n}$ of tensor products $\mathbf{e}_i^{\odot \lambda} = \mathbf{e}_{i_1}^{\odot \lambda_1} \odot \dots \odot \mathbf{e}_{i_m}^{\odot \lambda_m}$, indexed by $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{Y}$ and $\mathfrak{i} = (i_1, \dots, i_m) \in \mathbb{N}_*^m$ with $m = \ell(\lambda)$, uniquely defines the appropriate system

$$\varepsilon^{\mathbb{Y}} := \bigcup_{\lambda \in \mathbb{Y}} \{ \varepsilon_i^\lambda := \varepsilon_{i_1}^{\lambda_1} \odot \dots \odot \varepsilon_{i_m}^{\lambda_m} : \mathfrak{i} \in \mathbb{N}_*^m, m = \ell(\lambda) \}, \quad \varepsilon_i^\emptyset \equiv 1,$$

of χ -essentially bounded cylindrical functions in the variable $u \in \mathfrak{U}$ that possess continuous restrictions to \mathfrak{U}' .

Theorem 3.2. *For any $\mathfrak{i} \in \mathbb{N}_*^m$ and $\psi, \phi \in E_i^{\odot n}$, the following equality holds,*

$$\binom{n+m-1}{n} \int_{\mathfrak{U}} \phi_\zeta \bar{\psi}_\zeta d\chi = \langle \psi \mid \phi \rangle. \quad (3.5)$$

As a consequence, given $(\lambda, \mathfrak{i}) \in \mathbb{Y} \times \mathbb{N}_^m$ with $m = \ell(\lambda)$, the system $\varepsilon^{\mathbb{Y}}$ of functions ε_i^λ is orthogonal in the space L_X^2 and*

$$\| \varepsilon_i^\lambda \|_{L_X^2} = \left(\frac{(m-1)! \lambda!}{(m-1+|\lambda|)!} \right)^{1/2}. \quad (3.6)$$

Proof. Let E_i with $\mathfrak{i} = (i_1, \dots, i_m) \in \mathbb{N}_*^m$ be the m -dimensional subspace in E spanned by $\{\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_m}\}$ and $U(\mathfrak{i})$ be the unitary subgroup of $U(\infty)$ acting in E_i . The symbol $E_i^{\odot n}$ means the n th symmetric tensor power of E_i . Briefly

denote $\psi_{\dagger}[v\zeta(u)] := \langle ([v\rho^{-1}(u)]\mathbf{e}_1)^{\otimes n} | \psi \rangle$ with $\psi \in \mathbf{E}_i^{\odot n}$ for all $v \in U(i)$ and $u \in \mathfrak{U}$. Using (2.7) with $U(i)$ instead of $U(m)$, we have

$$\int_{\mathfrak{U}} \phi_{\dagger} \bar{\psi}_{\dagger} d\chi = \int_{\mathfrak{U}} d\chi(u) \int_{U(i)} \phi_{\dagger}[v\zeta(u)] \cdot \bar{\psi}_{\dagger}[v\zeta(u)] d\chi_i(v) \quad (3.7)$$

for all $\psi, \phi \in \mathbf{E}_i^{\odot n}$. It is clear that

$$\left| \int_{U(i)} \phi_{\dagger} \bar{\psi}_{\dagger} d\chi_i \right| \leq \sup_{v \in U(i)} |\phi_{\dagger}[v\zeta(u)]| |\psi_{\dagger}[v\zeta(u)]| \leq \|\phi\| \|\psi\|$$

for all $u \in \mathfrak{U}$. Hence, the corresponding sesquilinear form in (3.7) is continuous on $\mathbf{E}_i^{\odot n}$. Thus, there exists a linear bounded operator A over $\mathbf{E}_i^{\odot n}$ such that

$$\langle A\psi | \phi \rangle = \int_{U(i)} \phi_{\dagger} \bar{\psi}_{\dagger} d\chi_i.$$

Next we show that A commutes with all operators $w^{\otimes n} \in \mathcal{L}(\mathbf{E}_i^{\odot n})$ with $w \in U(i)$ acting as $w^{\otimes n} x^{\otimes n} = (wx)^{\otimes n}$, ($x \in \mathbf{E}_i$). Invariant properties (2.6) of χ_i under the right action (2.4) yield

$$\begin{aligned} & \langle (A \circ w^{\otimes n})\psi | \phi \rangle = \\ &= \int_{U(i)} \langle [v\zeta(u)]^{\otimes n} | \phi \rangle \overline{\langle [v\zeta(u)]^{\otimes n} | w^{\otimes n} \psi \rangle} d\chi_i(v) \\ &= \int_{U(i)} \langle [w^{-1}v\zeta(u)]^{\otimes n} | (w^{-1})^{\otimes n} \phi \rangle \overline{\langle [w^{-1}v\zeta(u)]^{\otimes n} | \psi \rangle} d\chi_i(v) \\ &= \int_{U(i)} \langle [v\zeta(u)]^{\otimes n} | (w^{-1})^{\otimes n} \phi \rangle \overline{\langle [v\zeta(u)]^{\otimes n} | \psi \rangle} d\chi_i(v) \\ &= \langle A\psi | (w^{-1})^{\otimes n} \phi \rangle = \langle (w^{\otimes n} \circ A)\psi | \phi \rangle, \end{aligned}$$

where $w^{-1} \in U(i)$ is the hermitian adjoint matrix of w . Hence, the equality

$$A \circ w^{\otimes n} = w^{\otimes n} \circ A, \quad w \in U(i) \quad (3.8)$$

holds. Let us check that the operator A , satisfying the condition (3.8), is proportional to the identity operator on $\mathbf{E}_i^{\otimes n}$. To this end we form the n th tensor power of the unitary group $U(i)$,

$$[U(i)]^{\otimes n} = \{w^{\otimes n} \in \mathcal{L}(\mathbf{E}_i^{\odot n}) : w \in U(i)\}, \quad [U(i)]^{\otimes 0} = 1.$$

Clearly, $[U(i)]^{\otimes n}$ is a unitary group over $\mathbf{E}_i^{\odot n}$. Let us check that the corresponding unitary representation

$$U(i) \ni w \longmapsto w^{\otimes n} \in \mathcal{L}(\mathbf{E}_i^{\odot n}) \quad (3.9)$$

is irreducible. This means that there is no subspace in $\mathbf{E}_i^{\odot n}$ other than $\{0\}$ and the whole space which is invariant under the action of $[U(i)]^{\otimes n}$.

Suppose, on the contrary, that there is an element $\psi \in \mathbf{E}_i^{\odot n}$ such that the equality $\langle ([w\rho^{-1}(u)]\mathbf{e}_1)^{\otimes n} | \psi \rangle = 0$ holds for all $w \in U(i)$ and $u \in U(\infty)$. By Lemma 3.1 the elements $w\rho^{-1}(u)$ act transitively on $\mathbf{S}(\infty)$. Hence, by n -homogeneity, we obtain $\langle x^{\otimes n} | \psi \rangle = 0$ for all $x \in \mathbf{E}_i$. Applying the polarization formula (3.3), we get $\psi = 0$. Hence, (3.9) is irreducible.

Thus, we can apply to (3.9) the Schur lemma [10, Thm 21.30]: a non-zero matrix which commutes with all matrices of an irreducible representation is a constant multiple of the unit matrix. As a result, we obtain that the operator A , satisfying (3.8), is proportional to the identity operator on $\mathbf{E}_i^{\odot n}$ i.e. $A = \alpha_{(n,i)} \mathbb{1}_{\mathbf{E}_i^{\odot n}}$ with a constant $\alpha_{(n,i)} > 0$. It follows that

$$\int_{U(i)} \phi_{\dagger} \bar{\psi}_{\dagger} d\chi_i = \alpha_{(n,i)} \langle \psi \mid \phi \rangle, \quad \phi, \psi \in \mathbf{E}_i^{\odot n}. \quad (3.10)$$

In particular, the subsystem of cylindrical functions ε_i^λ with a fixed $i \in \mathbb{N}_*^m$ is orthogonal in L_χ^2 , because the corresponding system of tensor products $\mathbf{e}_i^{\odot \lambda}$ indexed by $\lambda \in \mathbb{Y}_n$ with $\ell(\lambda) = m$ forms an orthogonal basis in $\mathbf{E}_i^{\odot n}$.

It remains to note that the set of all indices $i = (i_1, \dots, i_m) \in \mathbb{N}_*^m$ with all $m = \ell(\lambda)$ is directed with respect to the set-theoretic embedding, i.e., for any i, i' there exists i'' so that $i \cup i' \subset i''$. This fact and the above reasoning imply that the whole system $\varepsilon^\mathbb{Y}$ is also orthogonal in L_χ^2 .

Taking into account (3.2), we can choose $\phi_n = \psi_n = \varepsilon_i^\lambda \sqrt{n!/\lambda!}$ in (3.10). As a result, we obtain

$$\alpha_{(n,i)} = \frac{n!}{\lambda!} \int_{U(i)} |\varepsilon_i^\lambda|^2 d\chi_i = \frac{n!}{\lambda!} \|\varepsilon_i^\lambda\|_{L_\chi^2}^2.$$

The well known formula [18, 1.4.9] for the unitary m -dimensional group gives

$$\int_{U(i)} |\varepsilon_i^\lambda|^2 d\chi_i = \frac{\lambda!(m-1)!}{(n+m-1)!}, \quad |\lambda| = n, \quad \ell(\lambda) = m.$$

Using the last two formulas, we arrive at the relation

$$\alpha_{(n,i)} = \frac{n!}{\lambda!} \int_{U(i)} |\varepsilon_i^\lambda|^2 d\chi_i = \frac{n!}{\lambda!} \frac{\lambda!(m-1)!}{(n+m-1)!} = \frac{n!(m-1)!}{(n+m-1)!}. \quad (3.11)$$

Combining (3.7) and (3.11), we get (3.5) and, as a consequence, (3.6). \square

Definition 3.3. By H_χ^2 we denote the Hardy space over $U(\infty)$ defined as the L_χ^2 -closure of the complex linear span of the orthogonal system $\varepsilon^\mathbb{Y}$.

Let the space $H_\chi^{2,n}$ be the L_χ^2 -closure of the complex linear span of the subsystem $\varepsilon^{\mathbb{Y}_n} := \{\varepsilon_i^\lambda \in \varepsilon^\mathbb{Y} : (\lambda, i) \in \mathbb{Y}_n \times \mathbb{N}_*^{\ell(\lambda)}\}$ with a fixed $n \in \mathbb{Z}_+$.

Corollary 3.4. For any positive integers $n \neq k$ the orthogonality $H_\chi^{2,n} \perp H_\chi^{2,k}$ in L_χ^2 holds. As a consequence, the following orthogonal decomposition holds,

$$H_\chi^2 = \mathbb{C} \oplus H_\chi^{2,1} \oplus H_\chi^{2,2} \oplus \dots \quad (3.12)$$

Proof. The orthogonal property $\varepsilon_j^\mu \perp \varepsilon_i^\lambda$ with $|\mu| \neq |\lambda|$ for any $i \in \mathbb{N}_*^{\ell(\lambda)}$ and $j \in \mathbb{N}_*^{\ell(\mu)}$ follows from (2.8), since

$$\begin{aligned} \int_{\mathfrak{U}} \varepsilon_j^\mu \bar{\varepsilon}_i^\lambda d\chi &= \int_{\mathfrak{U}} \varepsilon_j^\mu (\exp(i\vartheta)u) \bar{\varepsilon}_i^\lambda (\exp(i\vartheta)u) d\chi(u) \\ &= \frac{1}{2\pi} \int_{\mathfrak{U}} \varepsilon_j^\mu \bar{\varepsilon}_i^\lambda d\chi \int_{-\pi}^{\pi} \exp(i(|\mu| - |\lambda|)\vartheta) d\vartheta = 0 \end{aligned}$$

for all $\lambda \in \mathbb{Y}$ and $\mu \in \mathbb{Y} \setminus \{\emptyset\}$. This yields $H_\chi^{2,|\mu|} \perp H_\chi^{2,|\lambda|}$ in the space L_χ^2 . \square

4. Reproducing kernels

Let us construct the reproducing kernel of H_χ^2 . We refer to [19] regarding reproducing kernels.

Lemma 4.1. *For every $u, v \in \mathfrak{U}$ there exists $q \in \mathbb{N}$ such that the reproducing kernel of the subspace $H_\chi^{2,n}$ in L_χ^2 has the form*

$$\begin{aligned} \mathfrak{h}_n(v, u) &= \sum_{m \leq q} \binom{n+m-1}{n} \langle \zeta(v) \mid \zeta(u) \rangle^n \\ &= \sum_{(\lambda, \iota) \in \mathbb{Y}_n \times \mathbb{N}_*^{\ell(\lambda)}} \frac{\varepsilon_i^\lambda(v) \bar{\varepsilon}_i^\lambda(u)}{\|\varepsilon_i^\lambda\|_{L_\chi^2}^2}, \quad u, v \in \mathfrak{U}. \end{aligned} \quad (4.1)$$

Proof. Note that $\mathfrak{h}_0 \equiv 1$. From (2.3) it follows that for each stabilized sequence $u \in \mathfrak{U}$ there exists $u_m \in U(m)$ with a certain $m = m(u)$ such that $u = \rho(u_m)$. So, the element $\zeta(u) = \rho^{-1}(u)\mathbf{e}_1$ is located on the m -dimensional sphere $S(m)$. It means that its Fourier series $\zeta(u) = \sum \mathbf{e}_k \varepsilon_k(u)$ has $m(u)$ terms. The tensor multinomial theorem yields the Fourier decomposition

$$[\zeta(u)]^{\otimes n} = \left(\sum \mathbf{e}_k \varepsilon_k(u) \right)^{\otimes n} = \sum_{(\lambda, \iota) \in \mathbb{Y}_n \times \mathbb{N}_*^{\ell(\lambda)}} \frac{n!}{\lambda!} \mathbf{e}_i^{\odot \lambda} \varepsilon_i^\lambda(u)$$

in the space $E^{\odot n}$. Using the formula (3.2), we obtain

$$\begin{aligned} \langle \zeta(v) \mid \zeta(u) \rangle^n &= \langle [\zeta(v)]^{\otimes n} \mid [\zeta(u)]^{\otimes n} \rangle \\ &= \sum_{(\lambda, \iota) \in \mathbb{Y}_n \times \mathbb{N}_*^{\ell(\lambda)}} \left(\frac{n!}{\lambda!} \right)^2 \langle \mathbf{e}_i^{\odot \lambda} \mid \mathbf{e}_i^{\odot \lambda} \rangle \varepsilon_i^\lambda(v) \bar{\varepsilon}_i^\lambda(u) = \sum_{(\lambda, \iota) \in \mathbb{Y}_n \times \mathbb{N}_*^{\ell(\lambda)}} \frac{\varepsilon_i^\lambda(v) \bar{\varepsilon}_i^\lambda(u)}{\|\varepsilon_i^\lambda\|_{L_\chi^2}^2} \end{aligned}$$

where $\langle \zeta(v) \mid \zeta(u) \rangle$ is decomposed into $q = \min\{m(u), m(v)\}$ summands in virtue of orthogonality. Multiplying both sides by $\binom{n+m-1}{n}$ and summing over all $m \leq q$, we get (4.1). It follows that $\int_{\mathfrak{U}} \mathfrak{h}_n(v, u) \varepsilon_i^\lambda(u) d\chi(u) = \varepsilon_i^\lambda(v)$ for each $v \in \mathfrak{U}$. Via Theorem 3.1 the system ε_i^λ of functions ε_i^λ forms an orthogonal basis in $H_\chi^{2,n}$. So, the integral operator

$$\int_{\mathfrak{U}} \mathfrak{h}_n(v, u) \psi_\zeta(u) d\chi(u) = \psi_\zeta(v), \quad \psi_\zeta \in H_\chi^{2,n} \quad (4.2)$$

acts identically on $H_\chi^{2,n}$. Thus, the kernel (4.1) is reproducing in $H_\chi^{2,n}$. \square

Let us consider the complex-valued kernel

$$\mathfrak{h}(z; v, u) = \prod_{m \leq \min\{m(u), m(v)\}} [1 - z \langle \zeta(v) \mid \zeta(u) \rangle]^{-m}, \quad u, v \in \mathfrak{U}, \quad |z| < 1$$

where $m(u)$ is the number of terms in the Fourier series $\zeta(u) = \sum \mathbf{e}_k \varepsilon_k(u)$.

Theorem 4.2. *The expansion $\mathfrak{h}(z; v, u) = \sum z^n \mathfrak{h}_n(v, u)$ holds for any $u, v \in \mathfrak{U}$ and $|z| < 1$. The kernel $\mathfrak{h}(1; v, u) = \sum \mathfrak{h}_n(v, u)$ is reproducing in H_χ^2 in the sense that*

$$\int_{\mathfrak{U}} \mathfrak{h}(1; v, u) f(u) d\chi(u) = f(v), \quad f \in H_\chi^2, \quad v \in \mathfrak{U}. \quad (4.3)$$

Proof. Let $q = \min\{m(u), m(v)\}$ and $m \leq q$. As is well known [18, 1.4.10],

$$[1 - z \langle \zeta(v) \mid \zeta(u) \rangle]^{-m} = \sum_{n \in \mathbb{Z}_+} \binom{n+m-1}{n} \langle z\zeta(v) \mid \zeta(u) \rangle^n \quad (4.4)$$

for all $|z| < 1$. By the Vandermonde identity, we have

$$\begin{aligned} \binom{n+m-1}{n} \langle z\zeta(v) \mid \zeta(u) \rangle^n &= \binom{r+k+p+l-2}{r+k} \langle z\zeta(v) \mid \zeta(u) \rangle^{r+k} \\ &= \sum_{r=0}^n \binom{r+p-1}{r} \binom{n-r+l-1}{n-r} \langle z\zeta(v) \mid \zeta(u) \rangle^{r+k} \end{aligned}$$

for all $n = r+k$ and $m = p+l-1$. Applying recursively this identity to the series (4.4) with any $m \leq q$ and using Lemma 4.1, we obtain

$$\begin{aligned} \mathfrak{h}(z; v, u) &= \prod_{m \leq q} \sum_{n \in \mathbb{Z}_+} \binom{n+m-1}{n} \langle z\zeta(v) \mid \zeta(u) \rangle^n \\ &= \sum_{n \in \mathbb{Z}_+} z^n \sum_{(\lambda, i) \in \mathbb{Y}_n \times \mathbb{N}_*^{\ell(\lambda)}} \frac{\varepsilon_i^\lambda(v) \bar{\varepsilon}_i^\lambda(u)}{\|\varepsilon_i^\lambda\|_{L_\chi^2}^2} = \sum_{n \in \mathbb{Z}_+} z^n \mathfrak{h}_n(v, u). \end{aligned}$$

Hence, the required expansion holds. By (3.12) we have $f = \sum_n f_n$ for any $f \in H_\chi^2$ where $f_n \in H_\chi^{2,n}$ is the orthogonal projection of f . Observing that $\mathfrak{h}_k(z; \cdot, u) \perp f_n(\cdot)$ with $n \neq k$ holds in L_χ^2 , we obtain

$$\int_{\mathfrak{U}} \mathfrak{h}(1; v, u) f(u) d\chi(u) = \sum \int_{\mathfrak{U}} \mathfrak{h}_n(v, u) f_n(u) d\chi(u) = \sum f_n(v) = f(v)$$

for all $v \in \mathfrak{U}$ and $f \in H_\chi^2$. Hence, (4.3) is valid. \square

5. The Hilbert-Schmidt analyticity

Recall (see e.g. [7]) that a function f on an open domain in a Banach space is said to be analytic if it is Gâteaux analytic and norm continuous. Similarly to [6, 15], we say that f is *Hilbert-Schmidt analytic* if its Taylor coefficients are Hilbert-Schmidt polynomials. Now we describe a space H^2 of Hilbert-Schmidt analytic complex functions on the open ball B .

The symmetric Fock space is defined to be the orthogonal sum

$$\Gamma = \bigoplus_{n \in \mathbb{Z}_+} E^{\odot n}, \quad \langle \psi \mid \phi \rangle = \sum_{n \in \mathbb{Z}_+} \langle \psi_n \mid \phi_n \rangle$$

for all elements $\psi = \bigoplus_n \psi_n$, $\phi = \bigoplus_n \phi_n \in \Gamma$ with $\psi_n, \phi_n \in \mathbf{E}^{\odot n}$. The subset $\{x^{\otimes n} : x \in \mathbf{B}\}$ is total in $\mathbf{E}^{\odot n}$ by virtue of (3.3). This provides the total property of the subsets $\{(1-x)^{-\otimes 1} : x \in \mathbf{B}\}$ in Γ where we denote

$$(1-x)^{-\otimes 1} := \sum x^{\otimes n}, \quad x^{\otimes 0} = 1.$$

The Γ -valued function $(1-x)^{-\otimes 1}$ in the variable $x \in \mathbf{B}$ is analytic, since

$$\|(1-x)^{-\otimes 1}\|^2 = \sum \|x\|^{2n} = (1 - \|x\|^2)^{-1} < \infty. \quad (5.1)$$

Let us define the Hilbert space of analytic complex functions in the variable $x \in \mathbf{B}$, associated with the Fock space Γ , as follows

$$H^2 = \{\psi^*(x) = \langle (1-x)^{-\otimes 1} \mid \psi \rangle : \psi \in \Gamma\}, \quad \|\psi^*\|_{H^2} := \|\psi\|$$

for all $x \in \mathbf{B}$. This description is correct, because each function ψ^* in the variable $x \in \mathbf{B}$ is analytic by virtue of [9, Prop. 2.4.2], as a composition of the analytic Γ -valued function $(1-x)^{-\otimes 1}$ in the variable $x \in \mathbf{B}$ and the linear functional $\langle \cdot \mid \psi \rangle$ on Γ .

Similarly, we define the closed subspace in H^2 of n -homogenous Hilbert-Schmidt polynomials ψ_n^* in the variable $x \in \mathbf{E}$ as

$$H_n^2 = \{\psi_n^*(x) = \langle x^{\otimes n} \mid \psi_n \rangle : \psi_n \in \mathbf{E}^{\odot n}\}.$$

Differentiating at zero any function $\psi^* = \bigoplus_n \psi_n^* \in H^2$ with $\psi_n^* \in H_n^2$, we obtain that its Taylor coefficients at zero $(n!)^{-1} d_0^n \psi^* = \psi_n^*$ are Hilbert-Schmidt polynomials. Hence, every function from H^2 is Hilbert-Schmidt analytic. Clearly, the following orthogonal decomposition holds,

$$H^2 = \mathbb{C} \oplus H_1^2 \oplus H_2^2 \oplus \dots \quad (5.2)$$

One can show that $(H_n^2)_n$ is a coherent sequence of polynomial ideals over \mathbf{E} in the meaning of [4, Def. 1.1].

For each pair $(\lambda, \iota) \in \mathbb{Y}_n \times \mathbb{N}_*^{\ell(\lambda)}$, we can uniquely assign the Hilbert-Schmidt n -homogenous polynomial

$$\hat{x}_\iota^\lambda := \langle x^{\otimes n} \mid \mathbf{e}_\iota^{\odot \lambda} \rangle, \quad x \in \mathbf{E},$$

defined via the Fourier coefficients $\hat{x}_k := \mathbf{e}_k^*(x) = \langle x \mid \mathbf{e}_k \rangle$ of an element $x \in \mathbf{E}$. Taking into account (3.2), the tensor multinomial theorem yields the following orthogonal decompositions with respect to the basis $\mathbf{e}^{\odot \mathbb{Y}}$ in Γ ,

$$(1-x)^{-\otimes 1} = \sum_{(\lambda, \iota) \in \mathbb{Y} \times \mathbb{N}_*^{\ell(\lambda)}} \frac{\hat{x}_\iota^\lambda \mathbf{e}_\iota^{\odot \lambda}}{\|\mathbf{e}_\iota^{\odot \lambda}\|^2}, \quad x \in \mathbf{B}. \quad (5.3)$$

Hence, any function $\psi^* \in H^2$ has the orthogonal expansion

$$\psi^*(x) = \langle (1-x)^{-\otimes 1} \mid \psi \rangle = \sum_{(\lambda, \iota) \in \mathbb{Y} \times \mathbb{N}_*^{\ell(\lambda)}} \hat{\psi}_{(\lambda, \iota)} \hat{x}_\iota^\lambda, \quad x \in \mathbf{B} \quad (5.4)$$

where $\hat{\psi}_{(\lambda, \iota)} := \langle \mathbf{e}_\iota^{\odot \lambda} \mid \psi \rangle \|\mathbf{e}_\iota^{\odot \lambda}\|^{-2}$ are the Fourier coefficients of $\psi \in \Gamma$ with respect to the basis $\mathbf{e}^{\odot \mathbb{Y}}$ and, moreover, $\|\psi^*\|_{H^2}^2 = \sum_{(\lambda, \iota)} |\langle \mathbf{e}_\iota^{\odot \lambda} \mid \psi \rangle|^2 \|\mathbf{e}_\iota^{\odot \lambda}\|^{-2}$. Thus, $\|\psi^*\|_{H^2}$ is a Hilbert-Schmidt type norm on H^2 .

6. Integral formulas

The one-to-one correspondence $\mathbf{e}_i^{\odot\lambda} \leftrightarrow \varepsilon_i^\lambda$ allows us to construct an antilinear isometric isomorphism $\mathcal{J}: \Gamma \rightarrow H_\chi^2$ and its adjoint $\mathcal{J}^*: H_\chi^2 \rightarrow \Gamma$ by the following change of orthonormal bases

$$\mathcal{J}: \Gamma \ni \mathbf{e}_i^{\odot\lambda} \|\mathbf{e}_i^{\odot\lambda}\|^{-1} \mapsto \varepsilon_i^\lambda \|\varepsilon_i^\lambda\|_{L_\chi^2}^{-1} \in H_\chi^2, \quad \lambda \in \mathbb{Y}, \quad i \in \mathbb{N}_*^{\ell(\lambda)}.$$

Clearly, $\mathcal{J}^*: \varepsilon_i^\lambda \|\varepsilon_i^\lambda\|_{L_\chi^2}^{-1} \mapsto \mathbf{e}_i^{\odot\lambda} \|\mathbf{e}_i^{\odot\lambda}\|^{-1}$, because $\langle \mathcal{J}\mathbf{e}_i^{\odot\lambda} | f \rangle_{L_\chi^2} = \langle \mathbf{e}_i^{\odot\lambda} | \mathcal{J}^*f \rangle$ for any $f \in H_\chi^2$. Using Theorem 3.2, for any element $\psi \in \Gamma$ with the Fourier coefficients $\hat{\psi}_{(\lambda,i)} = \langle \mathbf{e}_i^{\odot\lambda} | \psi \rangle \|\mathbf{e}_i^{\odot\lambda}\|^{-2}$, we obtain

$$\mathcal{J}\psi = \sum_{(\lambda,i) \in \mathbb{Y} \times \mathbb{N}_*^{\ell(\lambda)}} \hat{\psi}_{(\lambda,i)} \frac{\|\mathbf{e}_i^{\odot\lambda}\|^2}{\|\varepsilon_i^\lambda\|_{L_\chi^2}^2} \varepsilon_i^\lambda \quad \text{where} \quad \frac{\|\mathbf{e}_i^{\odot\lambda}\|^2}{\|\varepsilon_i^\lambda\|_{L_\chi^2}^2} = \frac{(\ell(\lambda) - 1 + |\lambda|)!}{(\ell(\lambda) - 1)!|\lambda|!}.$$

In particular, $\mathcal{J}x = \sum \hat{x}_k \varepsilon_k$ for any elements $x \in \mathbf{E}$ with the Fourier coefficients $\hat{x}_k = \langle x | \mathbf{e}_k \rangle$. Moreover, $\|\mathcal{J}x\|_{L_\chi^2}^2 = \sum \|\hat{x}_k\|^2 = \|x\|^2$.

In what follows, we assign to each $x \in \mathbf{E}$ the L_χ^2 -valued function

$$x_{\mathcal{J}}: \mathfrak{U} \ni u \mapsto (\mathcal{J}x)(u).$$

Lemma 6.1. *The function $\mathcal{J}(1-x)^{-\otimes 1} = (1-x_{\mathcal{J}})^{-1}$ in the variable $u \in \mathfrak{U}$ takes values in L_χ^2 for all $x \in \mathbf{B}$*

Proof. Applying \mathcal{J} to the decompositions (3.1) and (5.3), we obtain

$$\begin{aligned} \mathcal{J}(1-x)^{-\otimes 1} &= \sum_{(\lambda,i) \in \mathbb{Y} \times \mathbb{N}_*^{\ell(\lambda)}} \frac{\hat{x}_i^\lambda \varepsilon_i^\lambda}{\|\mathbf{e}_i^{\odot\lambda}\|^2} \\ &= \sum_{n \in \mathbb{Z}_+} \left(\sum_{k \in \mathbb{N}} \hat{x}_k \varepsilon_k \right)^n = (1-x_{\mathcal{J}})^{-1} \end{aligned} \tag{6.1}$$

where the following orthogonal series with a fixed $n \in \mathbb{N}$,

$$x_{\mathcal{J}}^n = \left(\sum_{k \in \mathbb{N}} \hat{x}_k \varepsilon_k \right)^n = \sum_{(\lambda,i) \in \mathbb{Y}_n \times \mathbb{N}_*^{\ell(\lambda)}} \frac{\hat{x}_i^\lambda \varepsilon_i^\lambda}{\|\mathbf{e}_i^{\odot\lambda}\|^2}, \tag{6.2}$$

is convergent in L_χ^2 . Moreover, taking into account the orthogonality, we get

$$\begin{aligned} \|(1-x_{\mathcal{J}})^{-1}\|_{L_\chi^2}^2 &= \sum_{n \in \mathbb{Z}_+} \sum_{(\lambda,i) \in \mathbb{Y}_n \times \mathbb{N}_*^{\ell(\lambda)}} \frac{|\hat{x}_i^\lambda|^2}{\|\mathbf{e}_i^{\odot\lambda}\|^2} \\ &= \sum_{n \in \mathbb{Z}_+} \left(\sum_{k \in \mathbb{N}} |\hat{x}_k|^2 \right)^n = (1-\|x\|^2)^{-1}. \end{aligned}$$

Hence, the function $(1-x_{\mathcal{J}})^{-1}$ with $x \in \mathbf{B}$ takes values in L_χ^2 . \square

Let $f = \sum_n f_n \in H_\chi^2$ with $f_n \in H_\chi^{2,n}$. Then $\mathcal{J}^*f \in \Gamma$ and $\mathcal{J}^*f_n \in \mathbf{E}^{\odot n}$. Briefly denote $\tilde{f} := (\mathcal{J}^*f)^* \in H_n^2$ and $\tilde{f}_n := (\mathcal{J}^*f_n)^* \in H^2$. Thus,

$$\tilde{f}(x) = \langle (1-x)^{-\otimes 1} | \mathcal{J}^*f \rangle, \quad x \in \mathbf{B},$$

$$\tilde{f}_n(x) = \langle x^{\otimes n} \mid \mathcal{J}^* f_n \rangle, \quad x \in \mathbf{E}.$$

Theorem 6.2. *Each Hilbert-Schmidt analytic function $\tilde{f} \in H^2$ has the integral representation*

$$\tilde{f}(x) = \int_{\mathfrak{U}} \frac{f d\chi}{1 - x_{\mathcal{J}}}, \quad x \in \mathbf{B} \quad (6.3)$$

and its Taylor coefficients at zero have the form

$$\frac{d_0^n \tilde{f}(x)}{n!} = \int_{\mathfrak{U}} x_{\mathcal{J}}^n f_n d\chi, \quad x \in \mathbf{E}. \quad (6.4)$$

The mapping $f \mapsto \tilde{f}$ produces the linear isometry $H_{\chi}^2 \simeq H^2$.

Proof. Consider the Fourier decomposition of f with respect to the basis $\varepsilon^{\mathbb{Y}}$ and its \mathcal{J}^* -image, respectively

$$f = \sum_{(\lambda, i) \in \mathbb{Y} \times \mathbb{N}_*^{\ell(\lambda)}} \hat{f}_{(\lambda, i)} \varepsilon_i^{\lambda}, \quad \mathcal{J}^* f = \sum_{(\lambda, i) \in \mathbb{Y} \times \mathbb{N}_*^{\ell(\lambda)}} \tilde{f}_{(\lambda, i)} \frac{\|\varepsilon_i^{\lambda}\|_{L_{\chi}^2}^2}{\|\mathbf{e}_i^{\odot \lambda}\|^2} \mathbf{e}_i^{\odot \lambda}$$

where $\hat{f}_{(\lambda, i)} = \|\varepsilon_i^{\lambda}\|_{L_{\chi}^2}^{-2} \int_{\mathfrak{U}} f \bar{\varepsilon}_i^{\lambda} d\chi$. Substituting $\hat{f}_{(\lambda, i)}$ to $\tilde{f} = (\mathcal{J}^* f)^*$ and using the orthogonal property and the relations (5.3) and (6.1), we obtain

$$\begin{aligned} \tilde{f}(x) &= \sum_{(\lambda, i) \in \mathbb{Y} \times \mathbb{N}_*^{\ell(\lambda)}} \frac{\hat{f}_{(\lambda, i)} \hat{x}_i^{\lambda} \langle \mathbf{e}_i^{\odot \lambda} \mid \mathbf{e}_i^{\odot \lambda} \rangle \|\varepsilon_i^{\lambda}\|_{L_{\chi}^2}^2}{\|\mathbf{e}_i^{\odot \lambda}\|^4} \\ &= \int_{\mathfrak{U}} \sum_{(\lambda, i) \in \mathbb{Y} \times \mathbb{N}_*^{\ell(\lambda)}} \frac{\hat{x}_i^{\lambda} \varepsilon_i^{\lambda}}{\|\mathbf{e}_i^{\odot \lambda}\|^2} f d\chi = \int_{\mathfrak{U}} \frac{f d\chi}{1 - x_{\mathcal{J}}}. \end{aligned}$$

Hence, (6.3) holds. Using (6.2), we similarly obtain

$$\tilde{f}_n(x) = \langle x^{\otimes n} \mid \mathcal{J}^* f_n \rangle = \int_{\mathfrak{U}} x_{\mathcal{J}}^n f_n d\chi. \quad (6.5)$$

Taking into account (6.5) and the orthogonal decomposition (3.12), we get

$$\tilde{f}(\alpha x) = \langle (1 - \alpha x)^{-\otimes 1} \mid \mathcal{J}^* f \rangle = \sum \alpha^n \int_{\mathfrak{U}} x_{\mathcal{J}}^n f_n d\chi, \quad |\alpha| \leq 1. \quad (6.6)$$

Note that $\tilde{f}(\alpha x)$ is analytic in α for all $x \in \mathbf{B}$. Differentiating $\tilde{f}(\alpha x)$ at $\alpha = 0$ and using the n -homogeneity of derivatives, we obtain

$$\frac{d^n}{d\alpha^n} \sum \alpha^n \int_{\mathfrak{U}} x_{\mathcal{J}}^n f_n d\chi \Big|_{\alpha=0} = n! \int_{\mathfrak{U}} x_{\mathcal{J}}^n f_n d\chi.$$

Hence, the functions (6.4) coincide with the Taylor coefficients at zero of \tilde{f} .

Finally, since the image of $\varepsilon^{\mathbb{Y}}$ under \mathcal{J}^* coincides with $\mathbf{e}^{\odot \mathbb{Y}}$, the mapping $H_{\chi}^2 \ni f \mapsto \tilde{f} \in H^2$ is an isometry. \square

7. Radial boundary values

Using (6.3), for each $f = \sum_n f_n \in H_\chi^2$ with $f_n \in H_\chi^{2,n}$ we can rewrite (6.6) as

$$\tilde{f}(rx) = \langle (1 - rx)^{-\otimes 1} \mid \mathcal{J}^* f \rangle = \int_{\mathfrak{U}} \frac{f d\chi}{1 - rx_{\mathcal{J}}}, \quad x \in \mathbf{K}, \quad r \in [0, 1)$$

where $\mathbf{K} = \{x \in \mathbf{E} : \|x\| \leq 1\}$.

Theorem 7.1. *The integral transform $\mathcal{C}_r : f \mapsto \mathcal{C}_r[f]$, defined as*

$$\mathcal{C}_r[f](x) := \int_{\mathfrak{U}} \frac{f d\chi}{1 - rx_{\mathcal{J}}}, \quad x \in \mathbf{K}, \quad r \in [0, 1), \quad (7.1)$$

belongs to the space of bounded linear operators $\mathcal{L}(H_\chi^2, H^2)$. The radial boundary values of $\mathcal{C}_r[f] \in H^2$ are equal to $\tilde{f} \in H^2$ in the following sense:

$$\lim_{r \nearrow 1} \|\mathcal{C}_r[f] - \tilde{f}\|_{H^2} = 0. \quad (7.2)$$

Moreover, the following equality holds,

$$\|\tilde{f}\|_{H^2}^2 = \sup_{r \in [0, 1)} \|\mathcal{C}_r[f]\|_{H^2}^2. \quad (7.3)$$

Proof. Theorem 6.2 and (7.1) imply the equality $\mathcal{C}_r[f] = \sum r^n \tilde{f}_n$ for any $r \in [0, 1)$. By (5.2), we have $\tilde{f}_k \perp \tilde{f}_n$ as $n \neq k$ in H^2 . It follows that

$$\|\mathcal{C}_r[f]\|_{H^2}^2 = \left\| \sum r^n \tilde{f}_n \right\|_{H^2}^2 = \sum r^{2n} \|\tilde{f}_n\|_{H^2}^2 = \sum r^{2n} \|f_n\|_{L_\chi^2}^2,$$

since \mathcal{J}^* acts isometrically from $H_\chi^{2,n}$ onto the space $\mathbf{E}^{\odot n}$ which is antilinear isometric to H_n^2 by definition. Similarly, we obtain that

$$\|\mathcal{C}_r[f] - \tilde{f}\|_{H^2}^2 = \sum (r^{2n} - 1) \|f_n\|_{L_\chi^2}^2 \longrightarrow 0, \quad r \rightarrow 1.$$

Moreover, the Cauchy-Schwarz inequality implies that

$$\|\mathcal{C}_r[f]\|_{H^2}^2 \leq \frac{1}{(1 - r^2)^{1/2}} \left(\sum \|f_n\|_{L_\chi^2}^2 \right)^{1/2} = \frac{\|f\|_{L_\chi^2}}{(1 - r^2)^{1/2}}$$

for all $f \in H_\chi^2$. Hence, the operator \mathcal{C}_r belongs to $\mathcal{L}(H_\chi^2, H^2)$ for all $r \in [0, 1)$.

Finally, the equalities

$$\sup_{r \in [0, 1)} \|\mathcal{C}_r[f]\|_{H^2}^2 = \sup_{r \in [0, 1)} \sum r^{2n} \|\tilde{f}_n\|_{H^2}^2 = \sum \|\tilde{f}_n\|_{H^2}^2 = \|\tilde{f}\|_{H^2}^2$$

give the required formula (7.3). \square

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